

On the effects of a common-pool resource on cooperation among firms with linear technologies

E. Gutiérrez, N. Llorca, J. Sánchez-Soriano*

CIO and Department of Statistics, Mathematics and Computer Science
University Miguel Hernández of Elche, Spain

M. A. Mosquera

Department of Statistics and Operations Research
University of Vigo, Spain

Abstract

In this paper we study the effect that the external management of a limited (natural) resource such as carbon dioxide or water quotas has on the behaviour of firms in a given sector. To do this, we choose a model in which all firms have the same technology and this is lineal. In the analysis of the problem games in partition function form arise in a natural way. It is proved, under certain conditions, that stable allocations exist in both cases with certainty and uncertainty.

AMS classification: 90B30, 91A12.

JEL Classification: C71.

Keywords: cooperative TU-games in partition function form, linear production situations, common-pool resource.

1 Introduction

In this paper we introduce linear production situations in which there is a limited common-pool resource. It is managed by an external agent and is absolutely necessary to produce any product. This type of situation appears frequently in real-life situations related to natural resource management such as when the producers need to buy carbon dioxide, water or fish quotas or even to obtain public capital to invest in their firms. Imagine the case that the producers, due to a new environmental regulation, have to introduce a restriction concerning the emissions of greenhouse gases, for instance. Let us assume that the greenhouse gases quotas can be bought from an external agent, who has a limited amount, at a given market price. We wonder what the effect of this common-pool resource on cooperation among firms will be when they have linear production techniques.

*Corresponding author, e-mail: joaquin@umh.es

Linear production (*LP*) situations are situations where several producers own resource bundles. They can use these resources to produce various products via linear production techniques that are available to all producers. The goal of each producer is to maximize their profit, which equals the revenue of their products at the given market prices. These situations and corresponding cooperative games are introduced in Owen (1975), where it is shown that these games always have a non empty core by constructing a core-element via a related dual linear program. Gellekom et al. (2000) named the set of all the core-elements that can be found in this way, the ‘Owen set’, and they provide a characterization of this. More general are situations involving a countably infinite number of products that can be produced. Tijs et al. (2001) study relations between the Owen set and the core of these semi-infinite situations.

When a limited common-pool resource is introduced in an *LP* situation this leads to a linear production situation with a common-pool resource (*LPP* situation). Although it may intuitively seem that when introducing a small change in the *LP* model everything will work in a similar way, in this case it is not true because, for instance, the games that arise in these situations are partition function form games and the existence of stable allocations is not always guaranteed.

Models with a limited external common resource have generally been addressed, in the literature, from the non-cooperative perspective. However, as Hardin (1968) points out the so-called tragedy of commons, where the common pool resource is overused, can occur. This is the main reason why we have considered a cooperative perspective in our model. Driessen and Meinhardt (2001) use a classic cooperative game, defined from a non-cooperative one, in which the value of a coalition (group of producers) is obtained from a two-person game where the members of the coalition try to maximize their profits in the worst case. Funaki and Yamato (1999) provide a cooperative approach to a model of an economy with a common-pool resource, where the demands on it are additive. In our case, we are interested in addressing *LP* situations with a limited common-pool resource from a cooperative point of view, where the value of a coalition is determined by taking into account not only what the members of the coalition can do, but also what outsiders can do. Thus, our model implies the use of games in partition function form introduced by Thrall and Lucas (1963) as in Funaki and Yamato (1999). In their model the distribution of fish (common-pool resource) among fishermen is carried out in proportion to the amount of labor involved.

This paper tackles the problem of allocating the common-pool resource focused on *LPP* situations. As in Funaki and Yamato (1999), the core of the games in partition function form associated with these situations can be reduced to the core of a related game in characteristic function form. The amount of the common-pool resource available can play a crucial role in the analysis of the associated games. We distinguish two cases: when the cooperation of all producers enables the common-pool resource to be sufficient for them and when it is not sufficient. In the former case, we show that the core of the games in partition function form is non empty. In the latter situation, additional condi-

tions are needed to assure the non emptiness of the core. The analysis of both cases is carried out taking into account that the partition function is unknown because the producers do not know how the common-pool resource is to be assigned. Therefore, this problem is approached as one with uncertainty. The study, where the partition function is known exactly, is left for further research.

The paper is organized as follows. Section 2 contains basic concepts on cooperative transferable utility games. In Section 3 linear production situations with a common-pool resource are introduced. We show that if the common-pool resource is not a constraint for the production process, or if it is so only for the coalition of all producers, these games can be reduced to games in characteristic function form. In the first case, we can find allocations in the core of the corresponding games, but in the second one the core can be empty. We introduce a new concept for partitions to be partitionally stable, which allows us, in some sense, to extend the concept of the core. In Section 4 we assume that the producers do not know how the common-pool resource will be distributed. Therefore, we introduce common-pool resource games to deal with this uncertainty. Different points of view can be used to define these games, we study the two extremes: the optimistic and pessimistic. Section 5 concludes.

2 Preliminaries

Let N be a non empty finite set of agents who agree to coordinate their actions. A cooperative game in characteristic function form is an ordered pair (N, v) , where N is the set of players and $v : 2^N \rightarrow \mathbb{R}$ is the characteristic function with $v(\emptyset) = 0$. This function assigns to each group of players (coalition), $S \subset N$, the value $v(S)$ which represents what the members in S obtain when they cooperate jointly. In a transferable utility game (TU -game) it is assumed that the utility can be linearly transferred among agents.

A classic issue in cooperative game theory is how to distribute the profit generated by the cooperating players. One way to do this is to use allocations in the core of the game. The core, $C(v)$, of a characteristic function form game (N, v) , introduced by Gillies (1953), is the subset of vectors in \mathbb{R}^N satisfying

$$\begin{array}{ll} \text{(Efficiency)} & \sum_{i \in N} x_i = v(N), \text{ and} \\ \text{(Coalitional rationality)} & \sum_{i \in S} x_i \geq v(S), \text{ for all } S \subset N. \end{array}$$

Bankruptcy problems were first introduced by O'Neill (1982). A standard bankruptcy problem can be described by a triple (N, E, d) , where $N = \{1, \dots, n\}$ is the finite set of agents, $E \geq 0$ is the estate to be divided and $d \in \mathbb{R}_+^N$, the vector of claims, is such that $\sum_{i \in N} d_i \geq E$. To deal with a bankruptcy problem (N, E, d) we can derive a classical bankruptcy TU -game (N, v) , where the value of a coalition $S \subset N$ is given by

$$v(S) = \max\{E - \sum_{i \in N \setminus S} d_i, 0\},$$

and represents what is left for the players in S after the demands of the players in $N \setminus S$ have been satisfied. These games have a non empty core.

In order to deal with our model, we will need to consider games in partition function form introduced by Thrall and Lucas (1963), where the worth of a coalition S depends not only on what the members in S can do, but also on what outsiders do. These are cooperative games with externalities. Formally, let $\mathcal{P}(N)$ denote the set of all partitions of N and $P = \{S_1, \dots, S_k\}$ represents one of these partitions or coalition structures, where the coalitions S_1, \dots, S_k are disjoint and their union is N . The pair $(S|P)$ such that $S \in P$ is usually called an embedded coalition. A cooperative game in partition function form is defined by $(N, \mathcal{P}(N), \{V(\bullet|P)\}_{P \in \mathcal{P}(N)})$, where N is the set of players, $\mathcal{P}(N)$ denotes the set of all partitions of N and $V(S|P)$ with $S \in P$ is a real number that represents the profit that a coalition $S \subseteq N$ can obtain when P is formed. Note that the profit that a coalition can obtain depends on the coalitions formed by the other players in $P \in \mathcal{P}(N)$.

Given a partition $P \in \mathcal{P}(N)$, a vector $x \in \mathbb{R}^n$ is said to be feasible under P if it satisfies $\sum_{i \in S} x_i \leq V(S|P), \forall S \in P$. We denote by \mathcal{F}^P the set of all feasible vectors under P and $\mathcal{F} = \cup_{P \in \mathcal{P}(N)} \mathcal{F}^P$ denotes the set of all feasible vectors. Given two vectors x, x' in \mathbb{R}^n , as in Funaki and Yamato (1999) we say that x dominates x' through S and denote $x \text{ dom}_S x'$ if the following conditions are satisfied:

1. $\sum_{i \in S} x_i \leq V(S|P), \forall P \in \mathcal{P}(N)$ such that $S \in P$,
2. $x_i > x'_i, \forall i \in S$.

We say that x dominates x' if there exists $S \subseteq N$ such that $x \text{ dom}_S x'$, and denote $x \text{ dom } x'$. The core of a cooperative game in partition function form is defined by $C(V) = \{x \in \mathcal{F} | \nexists x' \in \mathcal{F} \text{ s.t. } x' \text{ dom } x\}$. However, if we consider another definition of dominance, then we will obtain a different core. Thus, if we change condition 1 by

$$\text{I. } \sum_{i \in S} x_i \leq V(S|P), \text{ for some } P \in \mathcal{P}(N) \text{ with } S \in P,$$

we obtain a more restrictive concept of dominance that we denote by $\overline{\text{dom}}$ and the corresponding core is defined as $\overline{C}(V) = \{x \in \mathcal{F} | \nexists x' \in \mathcal{F} \text{ s.t. } x' \overline{\text{dom}} x\}$.

Associated with each game in partition function form two cooperative games in characteristic function form can be introduced: (N, v^-) and (N, v^+) , where

$$\begin{aligned} v^-(S) &= \min \{V(S|P) | P \in \mathcal{P}(N) \text{ such that } S \in P\}, \\ v^+(S) &= \max \{V(S|P) | P \in \mathcal{P}(N) \text{ such that } S \in P\}. \end{aligned}$$

(N, v^-) represents a pessimistic point of view of the gain that a coalition S can get, while (N, v^+) can be seen as optimistic. Funaki and Yamato (1999) proved that if $V(\{N\}|N) > \sum_{S \in P} V(S|P), \forall P \in \mathcal{P}(N)$, then

- a) $C(V) = C(v^-)$, and
- b) $\overline{C}(V) = C(v^+)$.

Given $P, P' \in \mathcal{P}(N)$, P' is a refinement of P if for all $S' \in P'$ there exists $S \in P$ such that $S' \subseteq S$, and it is denoted by $P' \subseteq P$. Using the concept of refinement an ordering of partitions arise in a natural way, with this ordering $(\mathcal{P}(N), \subseteq)$ is the so-called partition lattice.

3 Linear production situations with a common-pool resource

Let $N = \{1, \dots, n\}$ be a set of producers that face a linear production problem to produce a set $G = \{1, \dots, g\}$ of goods from a set $Q = \{1, \dots, q\}$ of resources. There exists an external common-pool resource, limited by an amount of r , that agents need to buy for producing the goods. The parameters of the model are:

- $b^i \in \mathbb{R}_+^q$ are the available resources for each producer $i \in N$, $b^S = \sum_{i \in S} b^i$. $B \in \mathcal{M}_{q \times n}$ is the resource matrix. We assume that there is a positive quantity available of each resource, that is, for all resources $t \in Q$ there is a producer i such that $b_i^t > 0$.
- The common pool-resource is not endowed to the producers but managed by an external agent. Its cost per unit is c and the total available is denoted by r .
- $A \in \mathcal{M}_{(q+1) \times g}$ is the production matrix, a_{tj} represents the amount of the resource t needed to produce the product j , where the last row is related to the common-pool resource and $a_{(q+1)j} > 0 \quad \forall j \in G$. Furthermore, we do not allow for output without input and therefore there exists at least one resource $t \in Q$ with $a_{tj} > 0 \quad \forall j \in G$.
- $p \in \mathbb{R}_{++}^g$ is the price vector. Moreover, in order to deal with a profitable process we assume that $p_j > a_{(q+1)j}c \quad \forall j \in G$.

Therefore, a linear production situation with a common-pool resource (*LPP*) can be represented by (A, B, p, r, c) .

To maximize his profit, producer i needs an optimal production plan $(x; z) \in \mathbb{R}_+^{g+1}$ that tells him how much he should produce of each good, x , and how much he needs of the common-pool resource, z . Not all production plans are feasible since the producer has to take into account his limited amount of resources. The amount of resources needed in a feasible production plan should not exceed the amount of resources available for producer i . Furthermore, a feasible production plan has to be nonnegative since we are only interested in producing nonnegative quantities of the products. The following linear program maximizes

the profit of producer i

$$\begin{aligned} \max \quad & \sum_{j=1}^g p_j x_j - cz \\ \text{s.t:} \quad & Ax \leq \begin{pmatrix} b^i \\ z \end{pmatrix} \\ & x \geq \mathbf{0}_g, z \geq 0. \end{aligned}$$

Thus, an optimal production plan for producer i is an optimal solution of this linear program. Apart from producing on their own, producers are allowed to cooperate. If a coalition S of producers cooperates then they put all their resources together and so given this amount of resources, the coalition wishes to maximize its profit,

$$\begin{aligned} \max \quad & \sum_{j=1}^g p_j x_j - cz \\ \text{s.t:} \quad & Ax \leq \begin{pmatrix} b^S \\ z \end{pmatrix} \\ & x \geq \mathbf{0}_g, z \geq 0. \end{aligned} \tag{1}$$

With an abuse of notation, we use z to represent the amount of the common-pool resource that a producer or a group of producers will need. We denote by $value(S; z)$ the value of this linear program, for every fixed z .

The optimal demand of the common-pool resource for each coalition S , $d_S = \min \{z \in \mathbb{R}_+ \mid value(S; z) \text{ is maximum}\}$, is obtained by solving the linear program (1). We should point out that these optimal demands are the desired amount of the common-pool resource for each coalition S and can be seen as their utopic or greatest aspirations a priori, before the common-pool resource is allocated. Note that they are not bounded from above by r .

Although it may seem that these demands are superadditive, i. e. $d_S \geq \sum_{i \in S} d_{\{i\}}$, this is not true as the next example shows.

Example 1 Let (A, B, r, p, c) be an LPP situation, with two producers, $N = \{1, 2\}$, who produce three products from two resources and a common-pool resource, where

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}, p = \begin{pmatrix} 4 \\ 4 \\ 8 \end{pmatrix}, c = 1, r = 5.$$

In this case, $d_{\{1\}} = d_{\{2\}} = 7$ while $d_{\{12\}} = 5$.

Next we present a technical result which guarantees that once we know that a positive profit is achieved, all the lower levels of the common-pool resource also provide positive profits.

Proposition 2 Let $S \subseteq N$, if there is z^* such that $value(S; z^*) > 0$, then $value(S; z) > 0$, for all z with $0 < z < z^*$.

Proof. Let z such that $0 < z < z^*$ and x^* the optimal solution corresponding to $value(S; z^*)$. Consider $x^z = \frac{z}{z^*} x^*$. The point $(x^z; z)$ is feasible for the problem

corresponding to S . Furthermore, it holds that:

$$\begin{aligned} \sum_{j=1}^g p_j x_j^* - cz^* > 0 &\implies \frac{z}{z^*} \left(\sum_{j=1}^g p_j x_j^* - cz^* \right) > 0 \implies \\ \sum_{j=1}^g p_j \left(\frac{z}{z^*} x_j^* \right) - c \left(\frac{z}{z^*} z^* \right) > 0 &\implies \sum_{j=1}^g p_j x_j^z - cz > 0. \end{aligned}$$

■

In the sequel, we will assume that for all S , there is a feasible production plan $(x; z)$ such that $value(S; z) > 0$. This implies that $d_S > 0$.

Let us assume that P is formed and, either through a collaborative procedure or through a competitive mechanism¹, the amount of the common-pool resource finally allocated to coalition $S \in P$ by the manager is $z_S(P)$. The profit that a coalition $S \subseteq N$ can obtain is given by

$$\begin{aligned} \max \quad & \sum_{j=1}^g p_j x_j - cz_S(P) \\ \text{s.t:} \quad & Ax \leq \begin{pmatrix} b^S \\ z_S(P) \end{pmatrix} \\ & x \geq \mathbf{0}_g. \end{aligned} \tag{2}$$

We should point out that $z_S(P)$ is bounded from above by r , because the manager of the common-pool resource cannot exceed this amount, while this does not hold for d_S , for all $S \subseteq N$.

Depending on the procedure used to obtain $z_S(P)$, which will be less or equal to its optimal demand d_S , we can define different games. These games are not characteristic function form games, but partition function form games and the amount available of the common-pool resource can play a crucial role on analysis.

Definition 3 Let (A, B, p, r, c) be an LPP situation. The partition function form game associated with this situation is given by $(N, \mathcal{P}(N), \{V(\bullet|P)\}_{P \in \mathcal{P}(N)})$, where N is the set of players, $\mathcal{P}(N)$ denotes the set of all partitions of N and $V(S|P)$ with $S \in P$ is obtained from the linear program (2), for all $S \subset N$, where $z_S(P)$ is the amount of common-pool resource available for coalition S when partition P is formed.

Proposition 4 Let (A, B, p, r, c) be an LPP situation and $(N, \mathcal{P}(N), \{V(\bullet|P)\}_{P \in \mathcal{P}(N)})$ the corresponding partition function form game. Then,

$$V(N|\{N\}) \geq \sum_{S \in P} V(S|P), \forall P \in \mathcal{P}(N).$$

Proof. Given $P \in \mathcal{P}(N)$, $V(S|P) = value(S; z_S(P))$, $\forall S \in P$ such that $\sum_{S \in P} z_S(P) \leq r$. Let be $(x^S; z_S(P))$ an optimal plan for each coalition $S \in P$.

¹At this moment we do not specify any process, what follows holds in any case.

Thus, $Ax^S \leq \begin{pmatrix} b^S \\ z_S(P) \end{pmatrix}$ and

$$A \begin{pmatrix} \sum_{S \in P} x^S \end{pmatrix} \leq \begin{pmatrix} \sum_{S \in P} b^S \\ \sum_{S \in P} z_S(P) \end{pmatrix} \leq \begin{pmatrix} b^N \\ r \end{pmatrix}.$$

Then, $\left(\sum_{S \in P} x^S; \sum_{S \in P} z_S(P) \right)$ is a feasible production plan for N and

$$\sum_{S \in P} \text{value}(S; z_S(P)) \leq \text{value} \left(N; \sum_{S \in P} z_S(P) \right) \leq V(N | \{N\}).$$

■

It is easy to check that $C(V)$ and $\overline{C}(V)$ only include efficient allocations. The following consequences are given without a proof because they can be derived in a similar manner as in Funaki and Yamato (1999).

Corollary 5 *Let (A, B, p, r, c) be an LPP situation, $(N, \mathcal{P}(N), \{V(\bullet | P)\}_{P \in \mathcal{P}(N)})$ the corresponding partition function form game and $(N, v^-), (N, v^+)$ the related games in characteristic function form. Then, $C(V) = C(v^-)$ and $\overline{C}(V) = C(v^+)$.*

If $(N, \mathcal{P}(N), \{V(\bullet | P)\}_{P \in \mathcal{P}(N)})$ is such that, $\forall S \subseteq N, \forall P \in \mathcal{P}(N)$ with $S \in P$, $V(S | P) = v(S)$, then the two definitions of dominance are equivalent, (N, v) is a game in characteristic function form and the core reduces to the well-known definition for games in characteristic function form. Next we show other two situations in which this holds.

Given a partition $P = \{S_1, \dots, S_k\}$, its total demand is $d(P) = \sum_{i=1}^k d_{S_i}$. An outstanding set associated with both the common-pool resource and the set of all partitions is the following:

$$M^{\min} = \{P \in \mathcal{P}(N) : d(P) > r \text{ and } P \text{ is minimal for the operator } \subseteq\},$$

where P is minimal for the operator \subseteq means that there is no refinement P' of P with $d(P') > r$.

The next results show that if the common-pool resource is not a constraint on the production process, i.e. $M^{\min} = \emptyset$, or it is only a restriction for the grand coalition, i.e. $M^{\min} = \{N\}$, then the corresponding partition function form games are characteristic function form games.

Proposition 6 *Let (A, B, p, r, c) be an LPP situation. If $M^{\min} = \emptyset$, then the corresponding game $(N, \mathcal{P}(N), \{V(\bullet | P)\}_{P \in \mathcal{P}(N)})$ is a characteristic function form game.*

Proof. If $M^{\min} = \emptyset$, then we have $d_N \leq r$. Thus, $z_S(P) = d_S$ for all $S \subseteq N$, due to $d(P) \leq r$ for all P . Therefore, for each $S \subseteq N$, $V(S|P) = V(S|P')$ for all $P, P' \in \mathcal{P}(N)$ such that $S \in P$, i.e. for each coalition the value does not depend on the coalitions formed by other players. ■

Proposition 7 *Let (A, B, p, r, c) be an LPP situation. If $\{N\} \in M^{\min}$, then $M^{\min} = \{N\}$ and the related game $(N, \mathcal{P}(N), \{V(\bullet|P)\}_{P \in \mathcal{P}(N)})$ is a characteristic function form game.*

Proof. If $\{N\} \in M^{\min}$ then, since $\{N\}$ is minimal for the operator \subseteq , $d_N > r$ and $d(P) \leq r$ for all $P \in \mathcal{P}(N)$, $P \neq \{N\}$. Similarly to Proposition 6, for each $S \subseteq N$, $V(S|P) = V(S|P')$ for all $P, P' \in \mathcal{P}(N)$, i.e. the value of coalition S does not depend on the coalitions formed by other players. On the other hand, the value of the grand coalition

$$\begin{aligned} \max \quad & \sum_{j=1}^g p_j x_j - cr \\ \text{s.t:} \quad & Ax \leq \begin{pmatrix} b^N \\ r \end{pmatrix} \\ & x \geq \mathbf{0}_g. \end{aligned} \tag{3}$$

only depends on its own, since there is no partition including N as a proper subset. Therefore, the related game is a characteristic function form game. ■

Let (A, B, p, r, c) be an LPP situation. The characteristic function form game associated with one of the two previous situations (N, v) , where $d_N \leq r$ or $d_N > r$ and $d(P) \leq r$ for all $P \neq \{N\}$, is given by $v(S) = \text{value}(S; z)$, with $z = d_S$ for all $S \neq N$ and $z = \min\{d_N, r\}$ for the grand coalition N . This is due to the fact that the common-pool resource is sufficient to satisfy the demands for all $S \neq N$, but for N the maximum amount available is r .

The next result shows that the characteristic function form game obtained when the common-pool resource is not a constraint for the production process has a non empty core.

Theorem 8 *Let (A, B, p, r, c) be an LPP situation with $M^{\min} = \emptyset$. The characteristic function form game (N, v) associated with this situation has a non empty core.*

Proof. The dual problem of (1)² for the grand coalition, N , is

$$\begin{aligned} \min \quad & \sum_{t=1}^q b_t^N y_t + 0y_{q+1} \\ \text{s.t:} \quad & A^t y \geq p \\ & y_{q+1} \leq c \\ & y \geq \mathbf{0}_{q+1}. \end{aligned} \tag{4}$$

²We use this problem because it is known by hypothesis that $\exists z \leq r$ for all problems, i.e. the common-pool resource is not scarce in any case.

An optimal solution of (1) for the grand coalition N is given by $(x^N; d_N)$ with $d_N \leq r$, and the related dual optimal solution is $(y_q^N; y_{q+1}^N)$, where with an abuse of notation from now on, we represent by y_q^N the vector (y_1^N, \dots, y_q^N) . From duality, it is known that $\sum_{j=1}^g p_j x_j^N - c d_N = \sum_{t=1}^q b_t^N y_t^N + 0 y_{q+1}^N = v(N)$. Therefore, somehow, the cost of the common-pool resource is charged to (discounted from) the value of the resources. It is easy to check that $(y_q^N; y_{q+1}^N)$ is feasible in the dual problem of (1) for every coalition $S \subset N$. Moreover, we have that for a dual optimal solution $(y_q^S; y_{q+1}^S)$ associated with the optimal solution $(x^S; d_S)$, it holds that $\sum_{t=1}^q b_t^S y_t^S + 0 y_{q+1}^S \geq \sum_{t=1}^q b_t^S y_t^S + 0 y_{q+1}^S = v(S)$. Thus, $\sum_{i \in S} (\sum_{t=1}^q b_t^i y_t^N + 0 y_{q+1}^N) \geq v(S), \forall S \subset N$, and this implies that $(b^i y^N)_{i \in N} \in C(v)$. \blacksquare

Corollary 9 *Let (A, B, p, r, c) be an LPP situation with $M^{\min} = \emptyset$, $(N, \mathcal{P}(N), \{V(\bullet|P)\}_{P \in \mathcal{P}(N)})$ the corresponding partition function form game and (N, v) the related game in characteristic function form. Then, $C(V) = \overline{C}(V) = C(v)$.*

Linear production (LP) situations and corresponding cooperative games were introduced in Owen (1975), where it is shown that these games have a nonempty core by constructing a core-element via a related dual linear program. Gellekom et al. (2000) named the set of all the core-elements that can be found in the same way as performed by Owen, the ‘Owen set’. Following this idea, we can introduce the Owen set of an LPP situation (A, B, p, r, c) , $Owen(A, B, p, r, c)$, as the set whose elements can be obtained through an optimal solution of (4) associated with the optimal solution of (1) $(x^N; d_N)$ such that $d_N \leq r$. To sum up, when $M^{\min} = \emptyset$ one way to obtain a stable distribution of the total profit is to use an element of the so-called Owen set of the LPP situation (A, B, p, r, c) . We should mention that this is similar to the classical results in the LP situations. However, it does not always work in the same way. Although the games such as those in Proposition 7 are characteristic function form games, they can have an empty core, as the following example shows.

Example 10 *Let (A, B, r, p, c) be an LPP situation, with three producers, $N = \{1, 2, 3\}$, who produce three products from three resources and a common-pool resource, where*

$$A = \begin{bmatrix} 3 & 6 & 6 \\ 6 & 6 & 6 \\ 5 & 10 & 6 \\ 2 & 4 & 4 \end{bmatrix}, B = \begin{bmatrix} 15 & 6 & 9 \\ 4 & 18 & 9 \\ 16 & 19 & 2 \end{bmatrix}, p = \begin{pmatrix} 10 \\ 9 \\ 9 \end{pmatrix}, c = 2, r = 10.$$

The demands are

$$d_{\{1\}} = \frac{4}{3}, d_{\{2\}} = 4, d_{\{3\}} = \frac{4}{5}, \\ d_{\{12\}} = \frac{22}{3}, d_{\{13\}} = \frac{17}{3}, d_{\{23\}} = \frac{42}{5}, d_N = \frac{31}{3},$$

and $\min\{\frac{31}{3}, 10\} = 10$. Since $M^{min} = \{N\}$ the corresponding TU-game (N, v) associated with this situation is given by

$$\begin{aligned} v(\{1\}) &= 4, v(\{2\}) = 12, v(\{3\}) = \frac{12}{5}, \\ v(\{12\}) &= 22, v(\{13\}) = 13, v(\{23\}) = \frac{126}{5}, v(N) = 30, \end{aligned}$$

and $C(v) = \emptyset$. Obviously, $C(V) = \overline{C}(V) = \emptyset$

Looking at this example one can observe that the partitions $\{N \setminus \{i\}, \{i\}\}_{i \in N}$ are the only stable in the following sense.

Definition 11 A partition $P \in \mathcal{P}(N)$ is said to be partitionally stable if the following two conditions hold $\forall S \in P$

(1) $C(v^S) \neq \emptyset$, and

(2) $\nexists \{T_k\}_{k=1}^l \in P$ such that $C\left(v^{S \cup \left\{\bigcup_{k=1}^l T_k\right\}}\right) \neq \emptyset$,

where (S, v^S) is the game reduced to coalition S when partition P is formed.

This definition of stability holds for games in partition function form (with whatever concept of dominance) and in characteristic function form and allows us, in some sense, to extend the concept of the core. Note that when the core of a game is non empty, then the grand coalition is the only one which is partitionally stable.

Proposition 12 Let (A, B, p, r, c) be an LPP situation with $\{N\} \in M^{min}$. The partitions $\{N \setminus \{i\}, \{i\}\}_{i \in N}$ are the only partitionally stable or the grand coalition if the core of the game is non empty.

Proof. If the core of the game is non empty the result holds. If it is empty, $d_{N \setminus \{i\}} + d_{\{i\}} \leq r, d_N > r$. The games associated with $\{N \setminus \{i\}\}_{i \in N}$ and $\{\{i\}\}_{i \in N}$ have a non empty core by Proposition 8 and any other partition does not satisfy condition (2) in the previous definition. ■

In the more general case, when the common-pool resource could be a constraint for the production process for some partition, each coalition of producers will obtain an amount of common-pool resource from either through a collaborative procedure or through a competitive mechanism. With respect to the information that they have on this process, we consider that they do not know the way in which r will be shared. The next section tackles this situation.

4 The common-pool resource game

In this section we assume that producers do not know how the common-pool resource is to be assigned. Therefore, they do not know the partition function form game V , so they face a problem under uncertainty. Thus, they can examine the problem of the amount they will receive from different points of view. We consider that what a coalition of producers S expects to receive from the common-pool resource can be described by the common-pool resource game (N, R) . These games are cooperative TU -games in characteristic function form and can be defined following different approaches. There are two extreme cases, depending on which point of view is used to deal with the situation, the optimistic and the pessimistic common-pool resource games, that will be addressed in this section. Hence, $R(S)$ is what coalition S thinks it can guarantee from the common-pool resource working on their own. It can be any value between those obtained from the optimistic and pessimistic points of view.

Once a coalition of producers S received its share of the common-pool resource, $R(S)$, using this amount as $z_S(P)$ in (2), for all $S \subseteq N$, the LPP game (N, v^R) is obtained, where $v^R(S) = \text{value}(S; R(S))$. In this way, the game associated with the LPP situation (A, B, r, p, c) obtained from the common-pool resource game (N, R) reduces to a characteristic function form game, (N, v^R) , since it does not depend on what the others may do.

The following theorem states a sufficient condition for the LPP game (N, v^R) to have a non empty core when $d_N > r$, no matter from what point of view the common-pool resource game (N, R) is defined. The case $d_N \leq r$ is studied in subsection 4.1.

Theorem 13 *Let (A, B, p, r, c) be an LPP situation, let (N, R) be the common-pool resource game associated with it, and (N, v^R) the corresponding LPP game. When $d_N > r$ if $C(R) \neq \emptyset$, then $C(v^R) \neq \emptyset$.*

Proof. Since $C(R) \neq \emptyset$, there is $u \in \mathbb{R}^N$ such that $u(S) = \sum_{i \in S} u_i \geq R(S)$, for all S , and $u(N) = r$. Let y^* be an optimal solution of the dual problem of (3). From duality theory, we know that $\sum_{t=1}^q b_t^N y_t^* + r y_{q+1}^* - cr = v^R(N)$. On the other hand, $\forall S \subseteq N$

$$\sum_{t=1}^q b_t^S y_t^* + u(S) y_{q+1}^* - cu(S) \geq \sum_{t=1}^q b_t^S y_t^* + R(S) (y_{q+1}^* - c) \geq v^R(S),$$

where the last inequality holds because y^* is feasible for the dual problem of coalition S and $y_{q+1}^* > c$ since $d_N > r$. Thus, $(b^i y^* + u_i (y_{q+1}^* - c))_{i \in N} \in C(v^R) \neq \emptyset$. ■

However, the opposite is not true in general as the next example shows.

Example 14 Let (A, B, r, p, c) be an LPP situation, with two producers, $N = \{1, 2\}$, who produce three products from two resources and a common-pool resource, where

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}, p = \begin{pmatrix} 4 \\ 4 \\ 8 \end{pmatrix}, c = 1, r = 4.$$

Consider the common-pool resource game, (N, R) , such that $R(\{1\}) = R(\{2\}) = R(\{12\}) = 4$. In this case, $C(R) = \emptyset$, $d_N = 5$ and $v^R(\{1\}) = v^R(\{2\}) = 10$, $v^R(N) = 28$, thus $C(v^R) \neq \emptyset$.

4.1 The optimistic approach

From an optimistic point of view, a coalition of producers S will obtain its demand. The related common-pool resource game (N, R^{opt}) is such that $R^{opt}(S) = \min\{d_S, r\}$.

Using the amount $R^{opt}(S)$ in (2), for all $S \subseteq N$, the optimistic LPP game (N, v^{opt}) is derived.

The core of this class of games can be non empty, as Example 14 illustrates. But $C(v^{opt}) = \emptyset$ on many occasions, as the next example shows.

Example 15 Let (A, B, r, p, c) be an LPP situation, with three producers, $N = \{1, 2, 3\}$, who produce two products from two resources and a common-pool resource, where

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 40 & 60 & 80 \\ 60 & 40 & 50 \end{bmatrix}, p = \begin{pmatrix} 50 \\ 60 \end{pmatrix}, c = 14, r = 50$$

and (N, v^{opt}) the related optimistic LPP game. In this case, the core of the optimistic LPP game will be all the points in \mathbb{R}^3 such that

$$\begin{aligned} x_1 &\geq 720, x_2 \geq 920, x_3 \geq 1150, \\ x_1 + x_2 &\geq 1640, x_1 + x_3 \geq 1936, x_2 + x_3 \geq 2070, x_1 + x_2 + x_3 = 2300, \end{aligned}$$

but it can be seen that there is no point satisfying all the above inequalities, then $C(v^{opt}) = \emptyset$. Taking into account that

$$\begin{aligned} d_{\{1\}} &= 20, d_{\{2\}} = 20, d_{\{3\}} = 25, \\ d_{\{12\}} &= 40, d_{\{13\}} = 46, d_{\{23\}} = 45, d_N = 66 \text{ and } \min\{66, 50\} = 50, \end{aligned}$$

it is easy to check that $C(R^{opt}) = \emptyset$.

We will study a situation in which $C(v^{opt})$ is non empty, but first we need some results. The following lemma tells us that when the common-pool resource is sufficient for the grand coalition, then the value of the grand coalition is an upper bound for the sum of the optimistic values in every partition.

Lemma 16 *Let (A, B, p, r, c) be an LPP situation and (N, v^{opt}) the LPP optimistic game associated with it. If $d_N \leq r$, then*

$$\sum_{S \in P} \text{value}(S, d_S) \leq \text{value}(N, d_N) = v^{opt}(N), \forall P \in \mathcal{P}(N).$$

Proof. We consider the linear program (1) for the grand coalition N :

$$\begin{aligned} \max \quad & \sum_{j=1}^g p_j x_j - cz \\ \text{s.t.:} \quad & Ax \leq \begin{pmatrix} b^N \\ z \end{pmatrix} \\ & x \geq \mathbf{0}_g, z \geq 0. \end{aligned} \tag{5}$$

Given $P \in \mathcal{P}(N)$, for every $S \in P$ there is an optimal solution $(x^S; d_S)$ for the linear program (1). Thus, $\left(\left(\sum_{S \in P} x_j^S \right)_{j=1}^g ; \sum_{S \in P} d_S \right)$ is a feasible solution for the linear program (5), therefore we have

$$\sum_{j=1}^g p_j \left(\sum_{S \in P} x_j^S \right) - c \left(\sum_{S \in P} d_S \right) \leq \sum_{j=1}^g p_j x_j^N - cd_N, \tag{6}$$

where $(x^N; d_N)$ is an optimal solution for (5) such that $d_N \leq r$. If we rewrite (6), we obtain $\sum_{S \in P} \sum_{j=1}^g (p_j x_j^S - cd_S) \leq \sum_{j=1}^g p_j x_j^N - cd_N$ and $\sum_{S \in P} \text{value}(S, d_S) \leq v^{opt}(N)$. \blacksquare

The following lemma, which is given without a proof because it is easy to derivate, give us two linear programs that, although they have different optimal solution sets, also have the same optimal values, i.e. they are optimally equivalents. Note that an optimal solution of the second one is the optimal demand of the common-pool resource for each coalition S , d_S . We should highlight that they only differ in a redundant constrain, $z \leq d_S$, however, this is the key with which to prove the next theorem.

Lemma 17 *Let (A, B, p, r, c) be an LPP situation. The following linear programs are optimally equivalents, for all S ,*

$$\begin{aligned} \max \quad & \sum_{j=1}^g p_j x_j - cz \\ \text{s.t.:} \quad & Ax \leq \begin{pmatrix} b^S \\ z \end{pmatrix} \\ & z \leq d_S \\ & x \geq \mathbf{0}_g, z \geq 0. \end{aligned} \tag{7}$$

$$\begin{aligned} \max \quad & \sum_{j=1}^g p_j x_j - cz \\ \text{s.t.:} \quad & Ax \leq \begin{pmatrix} b^S \\ z \end{pmatrix} \\ & x \geq \mathbf{0}_g, z \geq 0. \end{aligned} \tag{8}$$

The previous results provide us with two different, but equivalent, ways in which to tackle the linear programs. In the proof of the following theorem, we use one or the other depending on which will be more helpful. The next result tells us that cooperation eliminates the conflict. Because they could all go together to request the amount of the common-pool resource no matter which mechanism the manager uses to distribute it, regardless of what happens with the rest of partitions.

Theorem 18 *Let (A, B, p, r, c) be an LPP situation and (N, v^{opt}) the LPP optimistic game associated with it. If $d_N \leq r$, then $C(v^{opt}) \neq \emptyset$.*

Proof. Consider the linear program (7) for the grand coalition,

$$\begin{aligned} \max \quad & \sum_{j=1}^g p_j x_j - cz \\ \text{s.t:} \quad & Ax \leq \begin{pmatrix} b^N \\ z \end{pmatrix} \\ & z \leq d_N \\ & x \geq \mathbf{0}_g, z \geq 0. \end{aligned} \tag{9}$$

its dual is given by

$$\begin{aligned} \min \quad & \sum_{t=1}^q b_t^N y_t + 0y_{q+1} + d_N y_{q+2} \\ \text{s.t:} \quad & A^t y \geq p \\ & y_{q+1} - y_{q+2} \leq c \\ & y \geq \mathbf{0}_{q+2}. \end{aligned} \tag{10}$$

Let $(x^N; d_N)$ and $(y_q^N, y_{q+1}^N, 0)$ be the primal and dual optimal solutions for (9) and (10), respectively with $d_N \leq r$ and $y_{q+2}^N = 0^3$. It is easy to check that $(y_q^N, y_{q+1}^N, 0)$ is a feasible solution for the dual problem of (7) for every coalition S . If $(y_q^S, y_{q+1}^S, y_{q+2}^S)$ is an optimal dual solution associated with $(x^S; d_S)$, it holds that $\sum_{t=1}^q b_t^S y_t^N + 0y_{q+1}^N + d_N y_{q+2}^N \geq \sum_{t=1}^q b_t^S y_t^S + 0y_{q+1}^S + d_S y_{q+2}^S = \text{value}(S, d_S) = v^{opt}(S)$. Therefore, $\sum_{i \in S} (b_i^i y_i^N) \geq v^{opt}(S)$, $\forall S \subseteq N$, and this implies that $(b^i y^N)_{i \in N} \in C(v^{opt})$. ■

The following result is given without a proof because it is straightforward, since $v^{opt} \geq v^+ \geq v^-$.

Corollary 19 *Let (A, B, p, r, c) be an LPP situation, $(N, \mathcal{P}(N), \{V(\bullet|P)\}_{P \in \mathcal{P}(N)})$ the corresponding partition function form game and $(N, v^+), (N, v^-)$ the related games in characteristic function form. If $d_N \leq r$, then*

1. $C(v^+) \neq \emptyset$ and $\overline{C}(V) \neq \emptyset$
2. $C(v^-) \neq \emptyset$ and $C(V) \neq \emptyset$.

³This is true because the common-pool resource is not scarce in this case.

Remark 20 Note that this theorem holds for all LPP games (N, v^R) , obtained from any common-pool resource game (N, R) associated with an LPP situation, since $v^{opt}(S) \geq v^R(S)$, for every coalition S .

This result is important for several reasons. Firstly, we have found a case in which the core of the optimistic game is non empty. Secondly, the Owen set is very easy to obtain. Thirdly, it shows that cooperation among all agents is important when it makes the common-pool resource not to be scarce. Finally, in this case no matter how (N, R) or $z_S(P)$ are.

At a first glance, it seems that an easy condition to assure that the core is empty, when $d_N > r$, could be $\exists P \in \mathcal{P}(N)$ such that $\sum_{S \in P} d_S > r$, however, is not true as the next example shows.

Example 21 Let (A, B, r, p, c) be the LPP situation described in Example 14. In this case, $d_{\{1\}} = d_{\{2\}} = 7$, $d_N = 5$, $v^{opt}(1) = v^{opt}(2) = 10$ and $v^{opt}(N) = 28$. Thus, there is a partition $P = \{\{1\}, \{2\}\}$ where $d_{\{1\}} + d_{\{2\}} > 4$ and the core is non empty. Therefore, the aforementioned condition does not guarantee that the core is empty.

When $d_N > r$ and, $\forall P \in \mathcal{P}(N)$, $\sum_{S \in P} d_S < r$ the core of the optimistic game can be empty as Example 10 shows. But it does not hold in general as the following example illustrates.

Example 22 Let (A, B, r, p, c) be an LPP situation, with three producers, $N = \{1, 2, 3\}$, who produce three products from three resources and a common-pool resource, where

$$A = \begin{bmatrix} 10 & 8 & 7 \\ 7 & 10 & 5 \\ 3 & 6 & 7 \\ 5 & 2 & 4 \end{bmatrix}, B = \begin{bmatrix} 9 & 6 & 8 \\ 5 & 18 & 6 \\ 17 & 13 & 3 \end{bmatrix}, p = \begin{pmatrix} 8 \\ 9 \\ 5 \end{pmatrix}, c = 1, r = 5.$$

The corresponding optimistic game is

$$\begin{aligned} v^{opt}(1) &= 1.5, & v^{opt}(2) &= 5.25, & v^{opt}(3) &= 3.5, \\ v^{opt}(12) &= 13.125, & v^{opt}(13) &= 7.7, & v^{opt}(23) &= 12.25, \end{aligned}$$

and $v^{opt}(N) = 17.5$. The demands are

$$\begin{aligned} d_{\{1\}} &= 1, & d_{\{2\}} &= 1.5, & d_{\{3\}} &= 1, \\ d_{\{12\}} &= 3.75, & d_{\{13\}} &= 2.2, & d_{\{23\}} &= 3.5, \end{aligned}$$

with $d_N > 5$ and the core is non empty.

Hence, when $d_N > r$ we have from Theorem 13 a condition which is sufficient for the non emptiness of the core. In general, it is not clear whether the core of the optimistic game is empty or not.

4.2 The pessimistic approach

From a pessimistic point of view, a coalition of producers S will receive what agents outside S leave using the partition that minimizes the remainder for S . This situation can be described as a common-pool resource game (N, R^{pes}) ,

$$\text{where } R^{pes}(S) = \min \left\{ \min_{P: S \in P} \left\{ \left(r - \sum_{\substack{T \in P \\ T \neq S}} d_T \right)_+ \right\}, d_S \right\}.$$

Using this amount $R^{pes}(S)$ as $z_S(P)$ in (2), for all $S \subseteq N$, the pessimistic LPP game (N, v^{pes}) is obtained.

When $d_N \leq r$ the core of this game is non empty since the core of the optimistic game is non empty and $v^{opt} \geq v^+ \geq v^- \geq v^{pes}$. However, when $d_N > r$ it can be empty as Example 10 shows and, therefore, $C(v^+) = \overline{C}(V) = C(v^-) = C(V) = \emptyset$. We should point out that in Example 10 the optimistic and pessimistic games coincide. The following result states a condition for the non emptiness of the core of the pessimistic game.

Theorem 23 *Let (A, B, p, r, c) be an LPP situation and (N, v^{pes}) the pessimistic LPP game associated with it. If $d_N > r$ and $\sum_{i \in N} d_i \geq r$, then $C(v^{pes}) \neq \emptyset$.*

Proof. Let $\{d_i\}_{i \in N}$ be the individual demands of agents in N . Consider the common-pool resource game (N, w) , where $w(S) = (r - \sum_{i \notin S} d_i)_+$ and $w(N) = r$.

We will distinguish two cases:

- a) If $\sum_{i \in N} d_i > r$, (N, w) is a standard bankruptcy game and, therefore, it has a non empty core. Then, an $u \in \mathbb{R}^N$ such that $u(N) = r$ exists and $u(S) \geq$

$$(r - \sum_{i \notin S} d_i)_+ \geq \min_{P: S \in P} \left\{ \left(r - \sum_{\substack{T \in P \\ T \neq S}} d_T \right)_+ \right\} \geq \min \left\{ \min_{P: S \in P} \left\{ \left(r - \sum_{\substack{T \in P \\ T \neq S}} d_T \right)_+ \right\}, d_S \right\} = R^{pes}(S).$$

Therefore, $C(R^{pes}) \neq \emptyset$ and using the same arguments as in Theorem 13, the result holds.

- b) If $\sum_{i \in N} d_i = r$, $d(S) = \sum_{i \in S} d_i \geq (r - \sum_{i \notin S} d_i)_+ \geq \min_{P: S \in P} \left\{ \left(r - \sum_{\substack{T \in P \\ T \neq S}} d_T \right)_+ \right\} \geq \min \left\{ \min_{P: S \in P} \left\{ \left(r - \sum_{\substack{T \in P \\ T \neq S}} d_T \right)_+ \right\}, d_S \right\} = R^{pes}(S).$

Thus, $C(R^{pes}) \neq \emptyset$ and, then, from Theorem 13 $C(v^{pes}) \neq \emptyset$. ■

When $d_N > r$ and $\sum_{i \in N} d_i < r$, the core of the pessimistic game can be empty as in Example 10 or it can be non empty as Example 22 illustrates, since if the core of the optimistic game is non empty, the core of the pessimistic one is also non empty. Thus, in this case we have obtained similar results to those applying the optimistic approach.

5 Concluding remarks

The model proposed in this paper is novel, arises from many real-life situations and contains important changes with respect to linear production situations that affect cooperation. In spite of a small change in the LP model we obtain in a natural way, different situations where the corresponding games are, in general, games in partition function form as opposed to games in characteristic function form. Moreover, contrary to LP games these games can have an empty core.

We should highlight the role that demand of the common-pool resource for the grand coalition, d_N , plays when tackling the problem. We have come across an interesting case where cooperation makes the common-pool resource not to be scarce, $d_N \leq r$, in which the core of the corresponding game is non empty. When the common-pool resource is not sufficient, $d_N > r$, additional conditions are needed to assure the non emptiness of the core. The study of this case when the partition function is known exactly could be addressed using different approaches: from a non-cooperative point of view, such as that used in Gutierrez et al (2015), through an auction mechanism or with bankruptcy techniques which we would like to give our attention to in future research.

Acknowledgement. Financial support from the Government of Spain and FEDER under projects MTM2011-23205, MTM2011-27731-C03, MTM2014-53395-C3-3-P and MTM2014-54199-P, from Fundación Séneca de la Región de Murcia through grant 19320/PI/14 and from Xunta de Galicia under the project INCITE09-207-064-PR are gratefully acknowledged.

References

- [1] Driessen, TSH. and Meinhardt, H (2001). (Average-)convexity of common-pool and oligopoly TU-games. *International Game Theory Review*, 3: 141-158.
- [2] Funaki, Y and Yamato, T (1999). The core of an economy with a common pool resource: A partition function form approach. *International Journal of Game Theory*, 28: 157-171.
- [3] Gellekom, JRG van, Potters, JAM, Reijnierse, JH, Tijs, SH and Engel, MC (2000). Characterization of the Owen Set of Linear Production Processes. *Games and Economic Behavior*, 32: 139–156.
- [4] Gillies, DB (1953). Some theorems on n-person games. PhD thesis, Princeton University, Princeton.
- [5] Gutiérrez, E, Llorca, N, Mosquera, M and Sánchez-Soriano, J (2015). Equilibria in a competitive model arising from linear production situations with a common-pool resource. Mimeo.

- [6] Hardin, G (1968) The tragedy of commons. *Science*, 162: 1243-1248.
- [7] O'Neill, B (1982). A Problem of Rights Arbitration from the Talmud. *Mathematical Social Sciences* 2: 345-371.
- [8] Owen, G (1975). On the core of linear production games. *Mathematical Programming* 9: 358-370.
- [9] Thrall, RM and Lucas, WF (1963) n-person games in partition function form. *Naval Research Logistic Quarterly* 10: 281-298.
- [10] Tijs, SH, Timmer, J, Llorca, N and Sánchez-Soriano, J (2001). The Owen Set and the Core of Semi-Infinite Linear Production Situations. *Semi-Infinite Programming: Recent Advances* (eds.Goberna and López), 365-386, Kluwer Academic Publishers, Dordrecht.